Probability Theory
2024/25 Period IIb
Instructor: Gilles Bonnet
Resit
8/7/2024
Duration: 2 hours

Name: ______________________

Student number: ______________________

This exam contains 5 problems. Enter all requested information on the top of this page.

Your answers should be written in this booklet. Avoid handing in extra paper. In case you hand in extra paper and/or write parts of your answers in non obvious places, mark this explicitly so that it is not missed out during the grading process.

Do not use red ink.

You are **not** allowed to use any electronic devices during the exam.

You are **not** allowed to use any books, lecture notes or handwritten notes.

Each answer must be **justified**, unless explicitly stated otherwise.

Do not write on the table below.

Problem	Points	Score
1	12	
2	13	
3	18	
4	20	
5	27	
Total:	90	

Satisfying the above instructions gives you 10 extra free points.

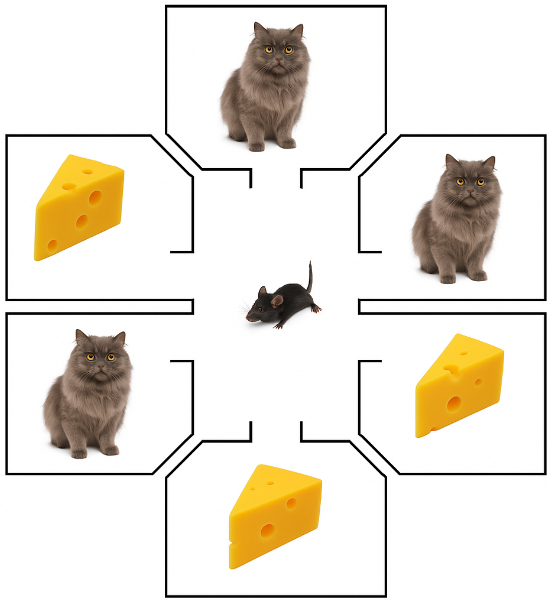
z	0	0,01	**0,02**	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0	0,5	0,50399	0,50798	0,51197	0,51595	0,51994	0,52392	0,5279	0,53188	0,53586
0,1	0,53983	0,5438	0,54776	0,55172	0,55567	0,55962	0,56356	0,56749	0,57142	0,57535
0,2	0,57926	0,58317	0,58706	0,59095	0,59483	0,59871	0,60257	0,60642	0,61026	0,61409
0,3	0,61791	0,62172	0,62552	0,6293	0,63307	0,63683	0,64058	0,64431	0,64803	0,65173
0,4	0,65542	0,6591	0,66276	0,6664	0,67003	0,67364	0,67724	0,68082	0,68439	0,68793
0,5	0,69146	0,69497	0,69847	0,70194	0,7054	0,70884	0,71226	0,71566	0,71904	0,7224
0,6	0,72575	0,72907	0,73237	0,73565	0,73891	0,74215	0,74537	0,74857	0,75175	0,7549
0,7	0,75804	0,76115	0,76424	0,7673	0,77035	0,77337	0,77637	0,77935	0,7823	0,78524
0,8	0,78814	0,79103	0,79389	0,79673	0,79955	0,80234	0,80511	0,80785	0,81057	0,81327
0,9	0,81594	0,81859	0,82121	0,82381	0,82639	0,82894	0,83147	0,83398	0,83646	0,83891
1	0,84134	0,84375	0,84614	0,84849	0,85083	0,85314	0,85543	0,85769	0,85993	0,86214
1,1	0,86433	0,8665	0,86864	0,87076	0,87286	0,87493	0,87698	0,879	0,881	0,88298
1,2	0,88493	0,88686	0,88877	0,89065	0,89251	0,89435	0,89617	0,89796	0,89973	0,90147
1,3	0,9032	0,9049	0,90658	0,90824	0,90988	0,91149	0,91309	0,91466	0,91621	0,91774
1,4	0,91924	0,92073	**0,9222**	0,92364	0,92507	0,92647	0,92785	0,92922	0,93056	0,93189
1,5	0,93319	0,93448	0,93574	0,93699	0,93822	0,93943	0,94062	0,94179	0,94295	0,94408
1,6	0,9452	0,9463	0,94738	0,94845	0,9495	0,95053	0,95154	0,95254	0,95352	0,95449
1,7	0,95543	0,95637	0,95728	0,95818	0,95907	0,95994	0,9608	0,96164	0,96246	0,96327
1,8	0,96407	0,96485	0,96562	0,96638	0,96712	0,96784	0,96856	0,96926	0,96995	0,97062
1,9	0,97128	0,97193	0,97257	0,9732	0,97381	0,97441	0,975	0,97558	0,97615	0,9767
2	0,97725	0,97778	0,97831	0,97882	0,97932	0,97982	0,9803	0,98077	0,98124	0,98169
2,1	0,98214	0,98257	0,983	0,98341	0,98382	0,98422	0,98461	0,985	0,98537	0,98574
2,2	0,9861	0,98645	0,98679	0,98713	0,98745	0,98778	0,98809	0,9884	0,9887	0,98899
2,3	0,98928	0,98956	0,98983	0,9901	0,99036	0,99061	0,99086	0,99111	0,99134	0,99158
2,4	0,9918	0,99202	0,99224	0,99245	0,99266	0,99286	0,99305	0,99324	0,99343	0,99361
2,5	0,99379	0,99396	0,99413	0,9943	0,99446	0,99461	0,99477	0,99492	0,99506	0,9952
2,6	0,99534	0,99547	0,9956	0,99573	0,99585	0,99598	0,99609	0,99621	0,99632	0,99643
2,7	0,99653	0,99664	0,99674	0,99683	0,99693	0,99702	0,99711	0,9972	0,99728	0,99736
2,8	0,99744	0,99752	0,9976	0,99767	0,99774	0,99781	0,99788	0,99795	0,99801	0,99807
2,9	0,99813	0,99819	0,99825	0,99831	0,99836	0,99841	0,99846	0,99851	0,99856	0,99861
3	0,99865	0,99869	0,99874	0,99878	0,99882	0,99886	0,99889	0,99893	0,99896	0,999
3,1	0,99903	0,99906	0,9991	0,99913	0,99916	0,99918	0,99921	0,99924	0,99926	0,99929
3,2	0,99931	0,99934	0,99936	0,99938	0,9994	0,99942	0,99944	0,99946	0,99948	0,9995
3,3	0,99952	0,99953	0,99955	0,99957	0,99958	0,9996	0,99961	0,99962	0,99964	0,99965
3,4	0,99966	0,99968	0,99969	0,9997	0,99971	0,99972	0,99973	0,99974	0,99975	0,99976
3,5	0,99977	0,99978	0,99978	0,99979	0,9998	0,99981	0,99981	0,99982	0,99983	0,99983
3,6	0,99984	0,99985	0,99985	0,99986	0,99986	0,99987	0,99987	0,99988	0,99988	0,99989
3,7	0,99989	0,9999	0,9999	0,9999	0,99991	0,99991	0,99992	0,99992	0,99992	0,99992
3,8	0,99993	0,99993	0,99993	0,99994	0,99994	0,99994	0,99994	0,99995	0,99995	0,99995
3,9	0,99995	0,99995	0,99996	0,99996	0,99996	0,99996	0,99996	0,99996	0,99997	0,99997
4	0,99997	0,99997	0,99997	0,99997	0,99997	0,99997	0,99998	0,99998	0,99998	0,99998

Table 1: CDF Φ of a standard normal distribution.
Example: the value in the row “1,4” and column “0,02” gives the approximation $\Phi(1.42) \simeq 0.9222$.

1. (12 points) A machine produces 6-sided dice. The machine is defective: while 99.9% of the dice it produces are normal, the remaining 0.1% have all their faces marked 6. Suppose I take (at random) a die produced by this machine and roll it n times, and then I inform you that all the rolls resulted in 6. For which values of n is it more likely that I took a defective die than that I took a normal die?

2. (13 points) The continuous random variables Z and W are independent, with Z following the exponential distribution with parameter 1 and W following the (continuous) uniform distribution on $(0, 1)$. Compute $\mathbb{P}(Z < W < 3Z)$.

3. A forgetful mouse is subjected to an experiment. It is placed inside the central room of a maze that has 7 rooms, 3 cats and 3 cheeses as shown below. The mouse always moves into a room chosen uniformly at random from all rooms adjacent to the room it is in, completely independently of all its previous choices.



The experimenters will only remove the mouse from the experiment if it's found all three cheeses. Of course, whenever our mouse enters a room with a cat it will play with the cat and will not leave that room ever again.

(a) (6 points) What is the probability the mouse will find at least one of the cheeses before entering a cat's room?

(b) (6 points) Given that the mouse has found one cheese before it meets the cat, what is the probability that the mouse will find a second cheese before entering a cat's room?

(c) (6 points) What is the chance the mouse will find the three cheeses before meeting a cat?

4. (20 points) Let $Z_1 \sim \mathcal{N}(0,1)$ and $Z_2 \sim \mathcal{N}(0,1)$ be independent standard normal random variables and

$$U = \sigma_1 Z_1 + \mu_1, \qquad V = \rho\sigma_2 Z_1 + \sqrt{1-\rho^2} \cdot \sigma_2 Z_2 + \mu_2.$$

Show that (U, V) follows a bivariate normal distribution (and write down the parameters of this distribution).

5. We have m urns and n balls, where $m \geq 2$ and n are integer numbers. We place the balls successively into the urns, so that any given ball is equally likely to go into any urn. Each placement is independent of the other ones.

   (a) (15 points) Let X and Y be the number of balls that go into urn 1 and 2, respectively. Compute $\mathrm{Cov}(X, Y)$.

   Hint. It can help to write X in the form $X = \sum_{i=1}^{n} X_i$, where X_i is the indicator that ball i goes into urn 1; and similarly for Y. Then you might want to compute $\mathrm{Cov}(X_i, Y_j)$ for $i \neq j$ and $i = j$ separately and use this to answer the question.

   (b) (6 points) Compute the variance of $X + Y$.

   (c) (6 points) Now, assume that there are $n = 2000$ balls and $m = 1000$ urns. Show that the probability that there are at least a total of 24 balls in total in the first two urns is less than 1%.

   Remark: If you show a (correct) stronger bound, this is obviously also fine.